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2004 J. Phys. A: Math. Gen. 37 2537

(http://iopscience.iop.org/0305-4470/37/7/002)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2537–2548 PII: S0305-4470(04)70515-6

The stochastic limit in the analysis of the open BCS model*

F Bagarello

Dipartimento di Matematica ed Applicazioni, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy

E-mail: bagarell@unipa.it

Received 14 October 2003
Published 4 February 2004
Online at stacks.iop.org/JPhysA/37/2537 (DOI: 10.1088/0305-4470/37/7/002)

Abstract

In this paper we show how the perturbative procedure known as *stochastic limit* may be useful in the analysis of the open BCS model discussed by Buffet and Martin as a spin system interacting with a fermionic reservoir. In particular we show how the same values of the critical temperature and of the order parameters can be found with a significantly simpler approach.

PACS numbers: 03.65.Yz, 02.50.Fz, 05.30.Jp

1. Introduction

In this paper we analyse the open BCS model as given in [1, 2] using the techniques of the *stochastic limit approach* (SLA), which is described in much detail in the monograph [3]. Instead of considering a fermionic reservoir, as the authors have done in [1, 2] (following the original suggestion contained in [4] which allows us to avoid dealing with unbounded operators), we will consider here a bosonic thermal bath. This choice is made to stay closer to the real physical world, where the reservoir is bosonic. This means that some of our formulae are only formal, but they can be made rigorous with just a little effort, using, for instance, the same framework for unbounded operators developed in [5] and references therein. We will comment again on this aspect of our model in the next section.

The main outcome of this paper is that the same values of the critical temperature and of the order parameters can be found using the SLA, in a significantly simpler way, as we will show in section 3. This simplification allows us to focus our attention on some aspects of the model which could appear not so clearly using the standard technique. This is what has been already observed in other physical applications: for instance, in [6], we used the SLA to explore in details some relations between different models of matter interacting with the radiation, such as the Hepp–Lieb and the Alli–Sewell models. Also, in [7], the SLA was used

* This paper is dedicated to my father.

in connection with the fractional quantum Hall effect, giving some interesting results. Other applications are contained in [3] and references therein.

The paper is organized as follows: in the next section we introduce the model and compute its generator using the SLA together with a semiclassical approximation, already introduced in [2], useful to obtain the free evolution of the matter operators; in section 3 we write the equations of motion for some macroscopic variables of the matter and we recover the same results as in [1]; our conclusions are presented in section 4, while the appendix is devoted to reviewing some facts concerning the SLA, useful to keep the paper self-contained.

2. The physical model and its stochastic limit

Our model consists of two main ingredients, the *system*, which is described by spin variables, and the *reservoir*, which is given in terms of bosonic operators. It is contained in a box of volume $V = L^3$, with N lattice sites. We define, following [1, 2],

$$H_N^{(\mathrm{sys})} = \tilde{\epsilon} \sum_{j=1}^{N} \sigma_j^0 - \frac{g}{N} \sum_{i,j=1}^{N} \sigma_i^+ \sigma_j^- \tag{2.1}$$

where the indices i, j represent discrete values of the momentum that an electron in a fixed volume can have, σ_j^+ creates a Cooper pair with given momentum while σ_j^- annihilates the same pair, $\tilde{\epsilon}$ is the energy of a single electron and $-g < 0$ is the interaction close to the Fermi surface. As we can see, only the $\pm$ component of the spin, that is the x, y components, have a mean field interaction, while the z component interacts with a constant external magnetic field. The algebra of the Pauli matrices is given by

$$\left[\sigma_i^+, \sigma_j^-\right] = \delta_{ij} \sigma_i^0 \qquad \left[\sigma_i^\pm, \sigma_j^0\right] = \mp 2 \delta_{ij} \sigma_i^\pm. \tag{2.2}$$

We will use the following realization of these matrices:

$$\sigma^0 \equiv \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If we now define the following operators,

$$S_N^\alpha = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^\alpha \qquad R_N = S_N^+ S_N^- = R_N^\dagger. \tag{2.3}$$

$H_N^{(\mathrm{sys})}$ can be simply written as $H_N^{(\mathrm{sys})} = N\left(\tilde{\epsilon} S_N^0 - g R_N\right)$, and it is easy to check that the following commutation rules hold

$$\left[S_N^0, R_N\right] = \left[H_N^{(\mathrm{sys})}, R_N\right] = \left[H_N^{(\mathrm{sys})}, S_N^0\right] = 0$$

for any given $N > 0$. It is also worth noting that the intensive operators S_N^α are all bounded by 1 in the operator norm, and that the commutators $\left[S_N^\alpha, \sigma_j^\beta\right]$ go to zero in norm as $\frac{1}{N}$ when $N \to \infty$, for all j, α and β.

As we have already mentioned in the introduction, we consider here a realistic bosonic reservoir, so that some of the following formulae must be understood to be *formal*. However, using, for instance, the same algebraic framework discussed in [5] for some different spin-boson models, or replacing the bosonic operators with their smeared versions, everything can be made rigorous. We avoid this useless complication here, since it would make all the treatment much more complicated, hiding in this way our main results.

Our construction of the reservoir follows the same steps given in [2], but for the commutation rules. We introduce here as many bosonic modes $a_{\vec{p},j}$ as lattice sites are present in V. This means that $j = 1, 2, \ldots, N$. $\vec{p}$ is the value of the momentum of the jth boson which, if we impose a periodic boundary condition on the wavefunctions, has necessarily the form $\vec{p} = \frac{2\pi}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ with $n_j \in \mathbb{Z}$. These operators satisfy the following canonical commutation relations (CCR),

$$[a_{\vec{p},i}, a_{\vec{q},j}] = \left[a^\dagger_{\vec{p},i}, a^\dagger_{\vec{q},j}\right] = 0 \qquad \left[a_{\vec{p},i}, a^\dagger_{\vec{q},j}\right] = \delta_{ij}\delta_{\vec{p}\vec{q}} \tag{2.4}$$

and their free dynamics is given by

$$H_N^{(\mathrm{res})} = \sum_{j=1}^{N} \sum_{\vec{p}\in\Lambda_N} \epsilon_{\vec{p}} a^\dagger_{\vec{p},j} a_{\vec{p},j}. \tag{2.5}$$

Here Λ_N is the set of values which $\vec{p}$ may take, according to the previous remark: $\Lambda_N = \left\{\vec{p} = \frac{2\pi}{L}\vec{n}, \vec{n} \in \mathbb{Z}^3\right\}$. It is useful to stress that the energy of the different bosons is clearly independent of the lattice site

$$\epsilon_{\vec{p}} = \frac{\vec{p}^2}{2m} = \frac{4\pi^2\left(n_1^2 + n_2^2 + n_3^2\right)}{2mL^2}.$$

The form of the interaction between reservoir and system is assumed to be of the following form:

$$H_N^{(I)} = \sum_{j=1}^{N} \left(\sigma_j^+ a_j(f) + \mathrm{h.c.}\right) \tag{2.6}$$

where we have introduced $a_j(f) = \sum_{\vec{p}\in\Lambda_N} a_{\vec{p},j} f(\vec{p})$, f being a given test function which will be asked to satisfy some extra conditions, see equation (2.24) below and the related discussion. We would like to stress that, in order to keep the notation reasonably simple, we will not use the tensor product symbol in this paper whenever the meaning of the symbols is clear.

The finite volume open system is now described by the following Hamiltonian,

$$H_N = H_N^0 + \lambda H_N^{(I)} \qquad \text{where} \qquad H_N^0 = H_N^{(\mathrm{sys})} + H_N^{(\mathrm{res})} \tag{2.7}$$

and λ is the coupling constant.

The first step in the SLA is the computation of the free evolution of the interaction Hamiltonian:

$$H_N^{(I)}(t) = \mathrm{e}^{\mathrm{i}H_N^0 t} H_N^{(I)} \mathrm{e}^{-\mathrm{i}H_N^0 t} = \sum_{j=1}^{N} \left(\mathrm{e}^{\mathrm{i}H_N^{(\mathrm{sys})}t} \sigma_j^+ \mathrm{e}^{-\mathrm{i}H_N^{(\mathrm{sys})}t} \mathrm{e}^{\mathrm{i}H_N^{(\mathrm{res})}t} a_j(f) \mathrm{e}^{-\mathrm{i}H_N^{(\mathrm{res})}t} + \mathrm{h.c.}\right). \tag{2.8}$$

The computation of the part of the reservoir is trivial and produces

$$\mathrm{e}^{\mathrm{i}H_N^{(\mathrm{res})}t} a_j(f) \mathrm{e}^{-\mathrm{i}H_N^{(\mathrm{res})}t} = a_j(f\,\mathrm{e}^{-\mathrm{i}t\epsilon})$$

where $a_j(f\mathrm{e}^{-\mathrm{i}t\epsilon}) = \sum_{\vec{p}\in\Lambda_N} a_{\vec{p},j} f(\vec{p})\,\mathrm{e}^{-\mathrm{i}t\epsilon_{\vec{p}}}$. This is an easy consequence of the CCR (2.4). The free evolution of the spin operators is more difficult and its expression can be found in [1, 2], for instance. Here it is shown how to obtain the time evolution in a *semiclassical* approximation, i.e. when the free time evolution of the intensive operators S_N^α is replaced by its limit (taken in the strong topology restricted to a certain family of relevant vectors [8]).

The differential equations of motion for the spin variables are

$$\begin{cases} \dfrac{\mathrm{d}\sigma_j^+(t)}{\mathrm{d}t} = 2\mathrm{i}\tilde{\epsilon}\sigma_j^+(t) + \mathrm{i}g S_N^+(t)\sigma_j^0(t) \\ \dfrac{\mathrm{d}\sigma_j^0(t)}{\mathrm{d}t} = 2\mathrm{i}g\left(\sigma_j^+(t)S_N^-(t) - \sigma_j^-(t)S_N^+(t)\right) \end{cases} \tag{2.9}$$

where we have called, with a little abuse of language which is quite useful to maintain the notation simple, $\sigma_j^\alpha(t) = e^{iH_N^{(sys)}t}\sigma_j^\alpha e^{-iH_N^{(sys)}t}$. In fact, to be more precise, instead of $\sigma_j^\alpha(t)$, we should write $\sigma_{j,N}^{\alpha,\mathrm{free}}(t)$, to stress the fact that $e^{iH_N^{(sys)}t}\sigma_j^\alpha e^{-iH_N^{(sys)}t}$ only produces the free evolution of σ_j^α, i.e. the evolution without any reservoir, and for N fixed. Moreover, in (2.9) we have introduced

$$S_N^\alpha(t) = e^{iH_N^{(sys)}t} S_N^\alpha e^{-iH_N^{(sys)}t} = \frac{1}{N}\sum_{j=1}^N e^{iH_N^{(sys)}t}\sigma_j^\alpha e^{-iH_N^{(sys)}t} = \frac{1}{N}\sum_{j=1}^N \sigma_j^\alpha(t).$$

Let us now call $S^\alpha = \mathcal{F}$-*strong* $\lim_{N\to\infty} S_N^\alpha$. The proof of the existence of this limit (together with all its powers) may be found in [8] and references therein. We can now take the sum over $j = 1, 2, \dots, N$ of (both sides of) the equations in (2.9), divide the result by N and consider the $\mathcal{F}$-*strong* $\lim_{N\to\infty}$ of the equations obtained in this way. We find that $\dot{S}^0(t) = 0$ and $\dot{S}^+(t) = i(2\tilde{\epsilon} + gS^0(t))S^+(t)$. These equations can be easily solved: $S^0(t) = S^0 = (S^0)^\dagger$ and $S^+(t) = S^+ e^{i(2\tilde{\epsilon}+gS^0)t}$. Of course, $S^-(t) = (S^+(t))^\dagger$. The system (2.9) gives now, if we replace $S_N^\alpha(t)$ with its $\mathcal{F}$-*strong* limit $S^\alpha(t)$,

$$\begin{cases} \dfrac{d\sigma_j^+(t)}{dt} = 2i\tilde{\epsilon}\sigma_j^+(t) + igS^+(t)\sigma_j^0(t) \\ \dfrac{d\sigma_j^0(t)}{dt} = 2ig\left(\sigma_j^+(t)S^-(t) - \sigma_j^-(t)S^+(t)\right). \end{cases} \tag{2.10}$$

This system is called the *semiclassical* approximation of (2.9), and it can be explicitly solved using, for instance, the Laplace transform technique. The computation is rather long, and we omit here all the details, which can be found in [1, 2]. Also, since only $\sigma_j^+(t)$ appear in (2.8), together with its Hermitian conjugate, we give here only the result we need. We have

$$\sigma_j^+(t) = e^{i\nu t}\rho_0^j + e^{i(\nu+\omega)t}\rho_+^j + e^{i(\nu-\omega)t}\rho_-^j \tag{2.11}$$

where we have defined the following operators

$$\begin{cases} \rho_0^j = \dfrac{g^2S^+}{\omega^2}\left(2S^-\sigma_j^+ + S^0\sigma_j^0 + 2S^+\sigma_j^-\right) \\ \rho_+^j = \dfrac{gS^+}{\omega^2}\left(gS^-\dfrac{\omega - gS^0}{\omega + gS^0}\sigma_j^+ + \dfrac{\omega - gS^0}{2}\sigma_j^0 - gS^+\sigma_j^-\right) \\ \rho_-^j = \dfrac{gS^+}{\omega^2}\left(gS^-\dfrac{\omega + gS^0}{\omega - gS^0}\sigma_j^+ - \dfrac{\omega + gS^0}{2}\sigma_j^0 - gS^+\sigma_j^-\right) \end{cases} \tag{2.12}$$

and the following quantities

$$\omega = g\sqrt{(S^0)^2 + 4S^+S^-} \qquad \nu = 2\tilde{\epsilon} + gS^0. \tag{2.13}$$

Defining further

$$\nu_\alpha(\vec{p}) = \nu - \epsilon_{\vec{p}} + \alpha\omega \tag{2.14}$$

where α takes the values 0, + and −, the operator $H_N^{(I)}(t)$ in (2.8) becomes

$$H_N^{(I)}(t) = \sum_{j=1}^N \sum_{\alpha=0,\pm} \left(\rho_\alpha^j a_j(f\, e^{it\nu_\alpha}) + \mathrm{h.c}\right). \tag{2.15}$$

Remark. It may be worth remarking that we would have obtained exactly this free time evolution even for a fermionic reservoir, since CCR and canonical anticommutation relations (CAR) produce the same free time evolution for both the annihilation and the creation operators.

From this point of view, the difference between a fermionic and a bosonic thermal bath appears really only a minor aspect of the model.

The next step in the SLA consists in computing the following quantity

$$I_\lambda(t) = \left(-\frac{\mathrm{i}}{\lambda}\right)^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \omega_{\mathrm{tot}} \left(H_N^{(I)}\left(\frac{t_1}{\lambda^2}\right) H_N^{(I)}\left(\frac{t_2}{\lambda^2}\right)\right) \tag{2.16}$$

and its limit for λ going to zero. Here the state ω_{tot} is the following product state $\omega_{\mathrm{tot}} = \omega_{\mathrm{sys}}\omega_\beta$, where ω_{sys} is a state of the system, while ω_β is a state of the reservoir, which we will take to be a Kubo–Martin–Schwinger (KMS) state corresponding to an inverse temperature $\beta = \frac{1}{kT}$. It is convenient here to use the so-called *canonical representation of thermal states* [3], which is sketched in the appendix. Then we introduce two sets of mutually commuting bosonic operators $\{c_{\vec{p},j}^{(\gamma)}\}$, $\gamma = a, b$, as follows:

$$a_{\vec{p},j} = \sqrt{m(\vec{p})}\, c_{\vec{p},j}^{(a)} + \sqrt{n(\vec{p})}\, c_{\vec{p},j}^{(b),\dagger} \tag{2.17}$$

where

$$m(\vec{p}) = \omega_\beta\big(a_{\vec{p},j} a_{\vec{p},j}^\dagger\big) = \frac{1}{1 - \mathrm{e}^{-\beta\epsilon_{\vec{p}}}} \qquad n(\vec{p}) = \omega_\beta\big(a_{\vec{p},j}^\dagger a_{\vec{p},j}\big) = \frac{\mathrm{e}^{-\beta\epsilon_{\vec{p}}}}{1 - \mathrm{e}^{-\beta\epsilon_{\vec{p}}}}. \tag{2.18}$$

The operators $c_{\vec{p},j}^{(\alpha)}$ satisfy the following commutation rules

$$\big[c_{\vec{p},j}^{(\alpha)}, c_{\vec{q},k}^{(\gamma)\dagger}\big] = \delta_{jk}\delta_{\vec{p}\vec{q}}\delta_{\alpha\gamma} \tag{2.19}$$

while all the other commutators are trivial. Furthermore, we introduce the vacuum of the operators $c_{\vec{p},j}^{(\alpha)}$, Φ_0:

$$c_{\vec{p},j}^{(\alpha)}\Phi_0 = 0 \qquad \forall \vec{p} \in \Lambda_N \qquad j = 1, \ldots, N \quad \alpha = a, b. \tag{2.20}$$

Finally, if we define $f_m(\vec{p}) = \sqrt{m(\vec{p})} f(\vec{p})$ and $f_n(\vec{p}) = \sqrt{n(\vec{p})} f(\vec{p})$, we get

$$a_j(f\,\mathrm{e}^{\mathrm{i}t\nu_\alpha}) = c_j^{(a)}(f_m\,\mathrm{e}^{\mathrm{i}t\nu_\alpha}) + c_j^{(b)\dagger}(f_n\,\mathrm{e}^{\mathrm{i}t\nu_\alpha}) \tag{2.21}$$

where we have used the usual following notation $c_j^{(a)}(g) = \sum_{\vec{p}\in\Lambda_N} c_{\vec{p},j}^{(a)} g(\vec{p})$ and $c_j^{(b)\dagger}(g) = \sum_{\vec{p}\in\Lambda_N} c_{\vec{p},j}^{(b)\dagger} g(\vec{p})$.[1] Therefore we have

$$H_N^{(I)}(t) = \sum_{j=1}^N \sum_{\alpha=0,\pm} \big\{\rho_\alpha^j \big(c_j^{(a)}(f_m\,\mathrm{e}^{\mathrm{i}t\nu_\alpha}) + c_j^{(b)\dagger}(f_n\,\mathrm{e}^{\mathrm{i}t\nu_\alpha})\big) + \mathrm{h.c}\big\} \tag{2.22}$$

and the KMS state ω_β can be represented as the following vector state, as in a Gelfand–Naimark–Segal (GNS)-like representation:

$$\omega_\beta(X_r) = \langle \Phi_0, X_r\Phi_0\rangle \tag{2.23}$$

for any observable of the reservoir, X_r, since ω_β is a Gaussian state [3]. This fact, together with (2.20) and with the commutation rules (2.19), simplifies the computation of the two point function $\omega_{\mathrm{tot}}\big(H_N^{(I)}\big(\frac{t_1}{\lambda^2}\big) H_N^{(I)}\big(\frac{t_2}{\lambda^2}\big)\big)$, which, after some algebraic computations, produces

$$\omega_{\mathrm{tot}}\left(H_N^{(I)}\left(\frac{t_1}{\lambda^2}\right) H_N^{(I)}\left(\frac{t_2}{\lambda^2}\right)\right) = \sum_{j=1}^N \sum_{\alpha,\beta=0,\pm} \sum_{\vec{p}\in\Lambda_N} \left\{\omega_{\mathrm{sys}}\big(\rho_\alpha^j \rho_\beta^{j\dagger}\big)|f_m(\vec{p})|^2 \exp\left(\mathrm{i}\frac{t_1}{\lambda^2}\nu_\alpha(\vec{p})\right)\right.$$
$$\times \exp\left(-\mathrm{i}\frac{t_2}{\lambda^2}\nu_\beta(\vec{p})\right) + \omega_{\mathrm{sys}}\big(\rho_\alpha^{j\dagger}\rho_\beta^j\big)|f_n(\vec{p})|^2$$
$$\left.\times \exp\left(-\mathrm{i}\frac{t_1}{\lambda^2}\nu_\alpha(\vec{p})\right)\exp\left(+\mathrm{i}\frac{t_2}{\lambda^2}\nu_\beta(\vec{p})\right)\right\}.$$

1 It may be worth noting that both $c_j^{(\gamma)}(f)$ and $c_j^{(\gamma)}(f)^\dagger$ are linear in their argument f.

Since we are interested in the limit $\lambda \to 0$ of $I_\lambda(t)$ we need to impose some conditions on the test function $f(\vec{p})$ [3]. In particular, we will require that the following finite integral exists:

$$\int_{-\infty}^{0} \mathrm{d}\tau \sum_{\vec{p}\in\Lambda_N} |f_r(\vec{p})|^2 \,\mathrm{e}^{\pm \mathrm{i}\tau\nu_\alpha(\vec{p})} < \infty \tag{2.24}$$

where $f_r(\vec{p})$ is $f_m(\vec{p})$ or $f_n(\vec{p})$ and $\nu_\alpha(\vec{p})$ is given in (2.14). Under this assumption we find that

$$I(t) = \lim_{\lambda\to 0} I_\lambda(t) = -t \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left\{ \omega_{\mathrm{sys}}\left(\rho_\alpha^j \rho_\alpha^{j\dagger}\right)\Gamma_\alpha^{(a)} + \omega_{\mathrm{sys}}\left(\rho_\alpha^{j\dagger}\rho_\alpha^j\right)\Gamma_\alpha^{(b)} \right\} \tag{2.25}$$

where the two complex quantities

$$\Gamma_\alpha^{(a)} = \int_{-\infty}^{0} \mathrm{d}\tau \sum_{\vec{p}\in\Lambda_N} |f_m(\vec{p})|^2 \,\mathrm{e}^{-\mathrm{i}\tau\nu_\alpha(\vec{p})} \qquad \Gamma_\alpha^{(b)} = \int_{-\infty}^{0} \mathrm{d}\tau \sum_{\vec{p}\in\Lambda_N} |f_n(\vec{p})|^2 \,\mathrm{e}^{\mathrm{i}\tau\nu_\alpha(\vec{p})} \tag{2.26}$$

both exist because of the assumption (2.24).

To this same result, we could also arrive starting with the following *stochastic limit Hamiltonian*

$$H_N^{(\mathrm{sl})}(t) = \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left\{ \rho_\alpha^j \left(c_{\alpha j}^{(a)}(t) + c_{\alpha j}^{(b)\dagger}(t) \right) + \mathrm{h.c} \right\} \tag{2.27}$$

where the operators $c_{\alpha j}^{(\gamma)}(t)$ are assumed to satisfy the following commutation rule,

$$\left[c_{\alpha j}^{(\gamma)}(t), c_{\beta k}^{(\mu)\dagger}(t')\right] = \delta_{jk}\delta_{\alpha\beta}\delta_{\gamma\mu}\delta(t-t')\Gamma_\alpha^{(\gamma)} \qquad \text{for} \quad t > t'. \tag{2.28}$$

We mean that, as is easily checked, the following quantity:

$$J(t) = (-\mathrm{i})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \Omega_{\mathrm{tot}}\left(H_N^{(\mathrm{sl})}(t_1) H_N^{(\mathrm{sl})}(t_2)\right)$$

coincides with $I(t)$. Here $\Omega_{\mathrm{tot}} = \omega_{\mathrm{sys}}\Omega = \omega_{\mathrm{sys}}\langle\Psi_0, \Psi_0\rangle$, where Ψ_0 is the vacuum of the operators $c_{\alpha j}^{(\gamma)}(t)$: $c_{\alpha j}^{(\gamma)}(t)\Psi_0 = 0$ for all α, j, γ and t [3].

Following the SLA [3], we now use $H_N^{(\mathrm{sl})}(t)$ to compute the generator of the theory. In fact, this is the main reason why this operator is introduced into the game. Let X be an observable of the system and $1\!\!1_r$ the identity of the reservoir. Its time evolution (after the stochastic limit is taken) is $j_t(X \otimes 1\!\!1_r) = U_t^\dagger (X \otimes 1\!\!1_r) U_t$, where U_t is the wave operator satisfying the following differential equation $\partial_t U_t = -\mathrm{i} H_N^{(\mathrm{sl})}(t) U_t$, whose adjoint is $\partial_t U_t^\dagger = \mathrm{i} U_t^\dagger H_N^{(\mathrm{sl})}(t)$.

Then we find

$$\begin{aligned}\partial_t j_t(X \otimes 1\!\!1_r) &= \mathrm{i} U_t^\dagger \left[H_N^{(\mathrm{sl})}(t), X \otimes 1\!\!1_r\right] U_t \\ &= \mathrm{i} U_t^\dagger \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left\{ \left[\rho_\alpha^j, X\right]\left(c_{\alpha j}^{(a)}(t) + c_{\alpha j}^{(b)\dagger}(t)\right) + \left[\rho_\alpha^{j\dagger}, X\right]\left(c_{\alpha j}^{(a)\dagger}(t) + c_{\alpha j}^{(b)}(t)\right) \right\} U_t.\end{aligned}$$

Next we have to normal order the formula above, i.e. to move to the right all the annihilation operators $c_{\alpha j}^{(\gamma)}(t)$ and to the left the creation operators $c_{\alpha j}^{(\gamma)\dagger}(t)$. To achieve this result we need to compute first the commutator $\left[c_{\alpha j}^{(a)}(t), U_t\right]$, and this can be done easily by means of the time consecutive principle [3], and of the commutation rules (2.28):

$$\begin{aligned}\left[c_{\alpha j}^{(a)}(t), U_t\right] &= -\mathrm{i} \int_0^t \left[c_{\alpha j}^{(a)}(t), H_N^{(\mathrm{sl})}(t')\right] U_{t'}\, \mathrm{d}t' \\ &= -\mathrm{i} \int_0^t \left(\rho_\alpha^{j\dagger} \Gamma_\alpha^{(a)} \delta(t-t')\right) U_{t'}\, \mathrm{d}t' = -\mathrm{i} \rho_\alpha^{j\dagger} \Gamma_\alpha^{(a)} U_t.\end{aligned} \tag{2.29}$$

Similarly we get

$$\left[c_{\alpha j}^{(b)}(t), U_t\right] = -\mathrm{i}\rho_\alpha^j \Gamma_\alpha^{(b)} U_t \tag{2.30}$$

and, taking the adjoint of these two equations,

$$\left[U_t^\dagger, c_{\alpha j}^{(a)\dagger}(t)\right] = \mathrm{i}U_t^\dagger \rho_\alpha^j \overline{\Gamma_\alpha^{(a)}} \qquad \text{and} \qquad \left[U_t^\dagger, c_{\alpha j}^{(b)\dagger}(t)\right] = \mathrm{i}U_t^\dagger \rho_\alpha^{j\dagger} \overline{\Gamma_\alpha^{(b)}}.$$

Going back to $\partial_t j_t(X \otimes \mathbb{1}_r)$ we find that

$$\begin{aligned}\partial_t j_t(X \otimes \mathbb{1}_r) = \mathrm{i}\sum_{j=1}^{N}\sum_{\alpha=0\pm} \Big\{&\big(\mathrm{i}U_t^\dagger \rho_\alpha^{j\dagger}\overline{\Gamma_\alpha^{(b)}} + c_{\alpha j}^{(b)\dagger}(t)U_t^\dagger\big)\big[\rho_\alpha^j, X\big]U_t \\ &+ \big(\mathrm{i}U_t^\dagger \rho_\alpha^{j}\overline{\Gamma_\alpha^{(a)}} + c_{\alpha j}^{(a)\dagger}(t)U_t^\dagger\big)\big[\rho_\alpha^{j\dagger}, X\big]U_t + U_t^\dagger\big[\rho_\alpha^j, X\big]\big(-\mathrm{i}\rho_\alpha^{j\dagger}\Gamma_\alpha^{(a)}U_t + U_t c_{\alpha j}^{(a)}(t)\big) \\ &+ U_t^\dagger\big[\rho_\alpha^{j\dagger}, X\big]\big(-\mathrm{i}\rho_\alpha^{j}\Gamma_\alpha^{(b)}U_t + U_t c_{\alpha j}^{(b)}(t)\big)\Big\}\end{aligned}$$

which has to be computed on the state Ω_{tot}. Therefore, since the generator L satisfies the equality $\Omega_{\mathrm{tot}}(\partial_t j_t(X \otimes \mathbb{1}_r)) = \Omega_{\mathrm{tot}}(j_t(L(X)))$, we get

$$L(X) = \sum_{j=1}^{N}\sum_{\alpha=0\pm}\left\{\big[\rho_\alpha^j, X\big]\rho_\alpha^{j\dagger}\Gamma_\alpha^{(a)} + \big[\rho_\alpha^{j\dagger}, X\big]\rho_\alpha^{j}\Gamma_\alpha^{(b)} - \rho_\alpha^j\big[\rho_\alpha^{j\dagger}, X\big]\overline{\Gamma_\alpha^{(a)}} - \rho_\alpha^{j\dagger}\big[\rho_\alpha^{j}, X\big]\overline{\Gamma_\alpha^{(b)}}\right\}. \tag{2.31}$$

This expression can be made simpler if the observable X is self-adjoint ($X = X^\dagger$). In this case we have

$$L(X) = L_1(X) + L_2(X) \tag{2.32}$$

where

$$\begin{aligned} L_1(X) &= \sum_{j=1}^{N}\sum_{\alpha=0\pm}\left\{\big[\rho_\alpha^j, X\big]\rho_\alpha^{j\dagger}\Gamma_\alpha^{(a)} + \text{h.c.}\right\} \\ L_2(X) &= \sum_{j=1}^{N}\sum_{\alpha=0\pm}\left\{\big[\rho_\alpha^{j\dagger}, X\big]\rho_\alpha^{j}\Gamma_\alpha^{(b)} + \text{h.c.}\right\}. \end{aligned} \tag{2.33}$$

This formula will be the starting point for the analysis in the next section.

Remark. Before going on, it may be interesting to stress that, when compared with the standard perturbative approach for the master equation for open quantum systems [2], the perturbative approach based on the SLA appears quite *friendly*. For instance, the so-called time consecutive principle and the new Hilbert space with ground vector Ψ_0 arising after the limit $\lambda \to 0$ is taken, are typical tools of the SLA and they are essential to make many computations almost trivial.

3. The phase transition

As discussed in [1, 2], S_N^0 and R_N are the relevant variables whose dynamics must be considered to analyse the phase structure of the model. These intensive operators are both self-adjoint, so that we can use equations (2.32) and (2.33) instead of (2.31). As a matter of fact, in both [1] and [2] these equations of motion are considered only as an intermediate step to compute the equation for $\Delta_N = \frac{1}{2}R_N^{1/2}$, which is called *the gap operator*. We will see in a while that the same conclusions as in [1, 2] can be obtained without introducing Δ_N but working directly with R_N and S_N^0.

As a first step we compute $L(S_N^0) = L_1(S_N^0) + L_2(S_N^0)$. We have, using (2.2), (2.3) and (2.33)

$$L_1(S_N^0) = \frac{1}{N}\sum_{j=1}^{N} L_1(\sigma_j^0) = \frac{1}{N}\sum_{j=1}^{N}\sum_{\alpha=0,\pm}\{[\rho_\alpha^j, \sigma_j^0]\rho_\alpha^{j\dagger}\Gamma_\alpha^{(a)} + \text{h.c.}\}$$

which can be written as

$$L_1(S_N^0) = \sum_{\alpha=0,\pm}\{(b_{\alpha+}S_N^+ + b_{\alpha-}S_N^- + b_{\alpha 0}S_N^0 + b_{\alpha 1}1\!\!1_N)\Gamma_\alpha^{(a)} + \text{h.c.}\} \tag{3.1}$$

where $1\!\!1_N = \frac{1}{N}\sum_{j=1}^{N} 1\!\!1_j$, and the various coefficients $\{b_{\alpha\gamma}\}$ have been introduced here only to stress the fact that $L_1(S_N^0)$ is linear in the intensive operators. As we have already mentioned before, the limit of the right-hand side of the formula exists in the strong topology restricted to a certain family $\mathcal{F}$ of states, since all the operators S_N^α converge in this topology. Therefore also the limit of the left-hand side does exist in the same topology. After some non-trivial algebra, we find

$$\begin{aligned} L_1(S^0) &:= \mathcal{F}\text{-}strong \lim_{N\to\infty} L_1(S_N^0) \\ &= -\frac{8g^4 S^0(S^+S^-)^2}{\omega^3}\left\{\text{Re}\,\Gamma_+^{(a)}\frac{\omega - g}{(\omega + gS^0)^2} + \text{Re}\,\Gamma_-^{(a)}\frac{\omega + g}{(\omega - gS^0)^2}\right\} \end{aligned} \tag{3.2}$$

where $\text{Re}\,\Gamma_\pm^{(a)}$ indicate the real part of $\Gamma_\pm^{(a)}$.

The computation of $L_2(S^0) := \mathcal{F}\text{-}strong \lim_{N\to\infty} L_2(S_N^0)$ follows essentially the same steps and produces

$$L_2(S^0) = -\frac{8g^4 S^0(S^+S^-)^2}{\omega^3}\left\{\text{Re}\,\Gamma_+^{(b)}\frac{\omega + g}{(\omega + gS^0)^2} + \text{Re}\,\Gamma_-^{(b)}\frac{\omega - g}{(\omega - gS^0)^2}\right\} \tag{3.3}$$

so that the final result is

$$L(S^0) = -\frac{8g^4 S^0(S^+S^-)^2}{\omega^3} h(S^0, S^+S^-). \tag{3.4}$$

Here we have introduced, for brevity, the function

$$\begin{aligned} h(S^0, S^+S^-) &= \text{Re}\,\Gamma_+^{(a)}\frac{\omega - g}{(\omega + gS^0)^2} + \text{Re}\,\Gamma_-^{(a)}\frac{\omega + g}{(\omega - gS^0)^2} \\ &\quad + \text{Re}\,\Gamma_+^{(b)}\frac{\omega + g}{(\omega + gS^0)^2} + \text{Re}\,\Gamma_-^{(b)}\frac{\omega - g}{(\omega - gS^0)^2} \end{aligned} \tag{3.5}$$

and we have made explicit the fact that h depends on $S^+S^- = \mathcal{F}\text{-}strong \lim_{N\to\infty} S_N^+S_N^-$ via the pulsation ω, see (2.13). It is interesting to observe that the same function $h(S^0, S^+S^-)$ appears in the computation of $L(S^+S^-) := \mathcal{F}\text{-}strong \lim_{N\to\infty} L(S_N^+S_N^-)$. Again, since $(S_N^+S_N^-)^\dagger = S_N^+S_N^-$, we can use formulae (2.32) and (2.33). Here the computations are significantly harder, but no difficulty of principle arises. As a technical tool it is convenient to use the fact that, in the limit $N \to \infty$, all the intensive operators commute with all the local operators of the system, $\lim_{N\to\infty}[S_N^\alpha, \sigma_j^\beta] = 0$, for all α, β and j. Therefore we get

$$L(S^+S^-) = -\frac{16g^4(S^+S^-)^3}{\omega^3} h(S^0, S^+S^-). \tag{3.6}$$

The phase structure of the model is now given by the right-hand sides of equations (3.4) and (3.6), see [1, 2], and, in particular, from the zeros of the functions

$$f_1(x, y) = -\frac{8g^4 xy^2}{\omega^3} h(x, y) \qquad f_2(x, y) = -\frac{16g^4 y^3}{\omega^3} h(x, y) \tag{3.7}$$

where we have introduced, to simplify the notation, $x = S^0$ and $y = S^+S^-$, so that $\omega = g\sqrt{x^2+4y}$ and $\nu = 2\tilde{\epsilon} + gx$. In particular, the existence of a superconducting phase corresponds to the existence of a non-trivial zero of f_1 and f_2 [1, 2]. Due to the definition of f_1 and f_2, it is clear that any (x_o, y_o), with $x_o \neq 0$ and $y_o \neq 0$, is such that $f_1(x_o, y_o) = f_2(x_o, y_o) = 0$ if and only if it is a zero of the function h: $h(x_o, y_o) = 0$. In order to find such a solution, it is first necessary to obtain an explicit expression for the coefficients $\mathrm{Re}\,\Gamma_\pm^{(\gamma)}$. This is easily done using the definitions in (2.26), since we get

$$\mathrm{Re}\,\Gamma_\pm^{(a)} = \frac{1}{2}\int_{-\infty}^{\infty}\sum_{\vec{p}\in\Lambda_N}|f_m(\vec{p})|^2\,\mathrm{e}^{-\mathrm{i}\tau\nu_\pm(\vec{p})}\,\mathrm{d}\tau = \pi\sum_{\vec{p}\in\Lambda_N}|f_m(\vec{p})|^2\delta(\nu_\pm(\vec{p})) \tag{3.8}$$

and

$$\mathrm{Re}\,\Gamma_\pm^{(b)} = \pi\sum_{\vec{p}\in\Lambda_N}|f_n(\vec{p})|^2\delta(\nu_\pm(\vec{p})). \tag{3.9}$$

It is now almost straightforward to recover the results of [1, 2]. Following Buffet and Martin's original idea, we look for solutions corresponding to $\nu = 0$. This means that, because of (2.13), the value of $x = S^0$ is fixed: $x = -2\tilde{\epsilon}/g$. Moreover, with this choice, $\nu_+(\vec{p}) = \omega - \epsilon_{\vec{p}}$, which is zero if and only if $\omega = \epsilon_{\vec{p}}$. Also, we have $\nu_-(\vec{p}) = -\omega - \epsilon_{\vec{p}}$, which is never zero. For these reasons we deduce that $\mathrm{Re}\,\Gamma_-^{(\gamma)} = 0$, $\gamma = a, b$, while the sums in (3.8) and (3.9) for $\mathrm{Re}\,\Gamma_+^{(\gamma)}$ are restricted to the smaller set, $\mathcal{E}_N \subset \Lambda_N$, of those values of $\vec{p}$ such that, if $\vec{q} \in \mathcal{E}_N$ then $\epsilon_{\vec{q}} = \omega$. Therefore, recalling the expressions for $m(\vec{p})$ and $n(\vec{p})$ in (2.18), we find

$$\mathrm{Re}\,\Gamma_+^{(a)} = \pi\frac{\mathrm{e}^{\beta\omega}}{\mathrm{e}^{\beta\omega}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2 \qquad \mathrm{Re}\,\Gamma_+^{(b)} = \pi\frac{1}{\mathrm{e}^{\beta\omega}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2. \tag{3.10}$$

From definition (3.5), therefore, we get the following equation

$$\pi\frac{\mathrm{e}^{\beta\omega}}{\mathrm{e}^{\beta\omega}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2\frac{\omega-g}{(\omega+gx)^2} + \pi\frac{1}{\mathrm{e}^{\beta\omega}-1}\sum_{\vec{p}\in\mathcal{E}_N}|f(\vec{p})|^2\frac{\omega+g}{(\omega+gx)^2} = 0$$

or

$$\mathrm{e}^{\beta\omega} = \frac{g+\omega}{g-\omega}. \tag{3.11}$$

This equation is the crucial one, which replaces the equation obtained in [1, 2], $g\tanh\left(\frac{\beta\omega}{2}\right) = \omega$. We conclude that

(1) First of all, introducing a new variable $\xi = \frac{\omega}{g}$, equation (3.11) has a non-trivial solution if and only if the function $g(\xi) = \mathrm{e}^{\beta g\xi} - \frac{1+\xi}{1-\xi}$ has a zero $\xi \neq 0$. It is clear that such a solution does exist only if the first derivative of $g(\xi)$, computed in $\xi = 0$, is positive, i.e. when $\beta g - 2 > 0$. This is because $g(0) = 0$ and $\lim_{\xi\to 1^-} g(\xi) = -\infty$. We recover therefore the first result of [1, 2], since this inequality implies the existence of a *critical temperature*, $T_c := \frac{g}{2k}$, coinciding with that found by Martin and Buffet, such that, when $T < T_c$, a $\tilde{\xi} \neq 0$ does necessarily exist such that $g(\tilde{\xi}) = 0$, and the system is in a superconducting phase.

(2) It is also possible to find the value of $y = S^+S^-$ directly from equation (3.11). However, in order to recover the same value of the gap operator known in the literature, we prefer to play a little bit with equation (3.11) in the following way:

$$g\tanh\left(\frac{\beta\omega}{2}\right) = g\frac{\mathrm{e}^{\frac{\beta\omega}{2}} - \mathrm{e}^{-\frac{\beta\omega}{2}}}{\mathrm{e}^{\frac{\beta\omega}{2}} + \mathrm{e}^{-\frac{\beta\omega}{2}}} = g\frac{\mathrm{e}^{\beta\omega}-1}{\mathrm{e}^{\beta\omega}+1} = g\frac{\frac{g+\omega}{g-\omega}-1}{\frac{g+\omega}{g-\omega}+1} = \omega.$$

This chain of equalities shows once again how our equation (3.11) returns the same equation obtained in [1, 2] with completely different techniques.

Vice versa, it is also straightforward to check that equation $g\tanh\left(\frac{\beta\omega}{2}\right) = \omega$ implies equation (3.11):

$$\mathrm{e}^{\beta\omega} = \frac{\tanh\left(\frac{\beta\omega}{2}\right)+1}{\tanh\left(\frac{\beta\omega}{2}\right)-1} = \frac{\frac{\omega}{g}+1}{1-\frac{\omega}{g}} = \frac{g+\omega}{g-\omega}$$

and this concludes the proof of the equivalence of our approach with that of Buffet and Martin.

4. Conclusions and comments

We have shown how the SLA can be successfully used to analyse the phase structure of low temperature superconductivity analysing a strong coupling BCS model, considered as an open system interacting with a bosonic thermal bath.

The procedure, which makes use of the canonical representation of thermal states, is rather direct and is technically much simpler than the one used in the original paper [1]. Among the other simplifications, for instance, a single equation $h(x, y) = 0$ must be solved instead of the system $f_1(x, y) = f_2(x, y) = 0$, which is the highly transcendental system which must be solved in [1].

For this reason we believe that it may be worth considering other models, still unsolved, with the simplifying strategy provided by the SLA, since new insights may eventually come out. For instance, one could first replace the bosonic reservoir with a reservoir made of quons, [9], in an attempt to get a different free time evolution for the creation and annihilation quon operators. Following our analysis, and in particular definition (3.5) of the function h, this is in fact the easiest way to get an higher value of the critical temperature $\left(T_{\mathrm{c}} > \frac{g}{2k}\right)$. Another possibility to achieve the same result is to consider a second reservoir interacting with the first one: again, in this way the free time evolution of the bosonic operators will be different from the one considered here, $a_j(f\,\mathrm{e}^{-\mathrm{i}\epsilon t})$. These models will be considered in a forthcoming paper [10].

Acknowledgment

This work is partly supported by MURST.

Appendix. A few results on the stochastic limit

In this appendix, we will briefly summarize some of the basic facts and properties concerning the SLA which are used throughout the paper. We refer to [3] and references therein for more details.

Given an open system $\mathcal{S}+\mathcal{R}$ we write its Hamiltonian H as the sum of two contributions, the free part H_0 and the interaction λH_I. Here λ is a coupling constant, H_0 contains the free evolution of both the system $\mathcal{S}$ and the reservoir $\mathcal{R}$, while H_I contains the interaction between $\mathcal{S}$ and $\mathcal{R}$. Working in the interaction picture, we define $H_I(t) = \mathrm{e}^{\mathrm{i}H_0t} H_I\,\mathrm{e}^{-\mathrm{i}H_0t}$ and the so-called wave operator $U_\lambda(t)$ which is the solution of the following differential equation

$$\partial_t U_\lambda(t) = -\mathrm{i}\lambda H_I(t) U_\lambda(t) \tag{A.1}$$

with the initial condition $U_\lambda(0) = \mathbb{1}$. Using the van Hove rescaling $t \to \frac{t}{\lambda^2}$, see [2, 3], for instance, we can rewrite the same equation in a form which is more convenient for our perturbative approach, that is

$$\partial_t U_\lambda\left(\frac{t}{\lambda^2}\right) = -\frac{\mathrm{i}}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) U_\lambda\left(\frac{t}{\lambda^2}\right) \tag{A.2}$$

with the same initial condition as before. Its integral counterpart is

$$U_\lambda\left(\frac{t}{\lambda^2}\right) = \mathbb{1} - \frac{\mathrm{i}}{\lambda}\int_0^t H_I\left(\frac{t'}{\lambda^2}\right) U_\lambda\left(\frac{t'}{\lambda^2}\right) \mathrm{d}t' \tag{A.3}$$

which is the starting point for a perturbative expansion, which works in the following way.

Suppose, to begin with, that we are interested in the zero temperature situation. Then let φ_0 be the ground vector of the reservoir and ξ be a generic vector of the system. Now we put $\varphi_0^{(\xi)} = \varphi_0 \otimes \xi$. We want to compute the limit, for λ going to 0, of the first non-trivial order of the mean value of the perturbative expansion of $U_\lambda(t/\lambda^2)$ above in $\varphi_0^{(\xi)}$, that is the limit of

$$I_\lambda(t) = \left(-\frac{\mathrm{i}}{\lambda}\right)^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left\langle H_I\left(\frac{t_1}{\lambda^2}\right) H_I\left(\frac{t_2}{\lambda^2}\right)\right\rangle_{\varphi_0^{(\xi)}} \tag{A.4}$$

for $\lambda \to 0$. Under some regularity conditions on the functions which are used to smear out the (typically) bosonic fields of the reservoir, this limit is shown to exist for many relevant physical models, see [3] and [7, 6] for a few recent applications to quantum many body theory. It is at this stage that all the complex quantities such as the $\Gamma_\alpha^{(\gamma)}$ we have introduced in the main body of this paper appear. We define $I(t) = \lim_{\lambda\to 0} I_\lambda(t)$. In the same sense of the convergence of the (rescaled) wave operator $U_\lambda\left(\frac{t}{\lambda^2}\right)$ (the convergence in the sense of correlators), it is possible to check that also the (rescaled) reservoir operators converge and define new operators which do not satisfy canonical commutation relations but a modified version of these. For instance, in section 2 this procedure has produced the operators $c_{\alpha j}^{(\gamma)}$ starting from $c_{\vec{p},j}^{(\gamma)}$. Moreover, these limiting operators depend explicitly on time and they live in a Hilbert space which is different from the original one. In particular, they annihilate a vacuum vector, η_0, which is no longer the original one, φ_0. This is what happens, for instance, if φ_0 depends on λ, $\varphi_0 \to \varphi_0^{(\lambda)}$, and considering η_0 as the following limit: $\eta_0 = \lim_{\lambda\to 0} \varphi_0^{(\lambda)}$.

It is not difficult to deduce the form of a time-dependent self-adjoint operator $H_I^{(\mathrm{sl})}(t)$, which depends on the system operators and on the limiting operators of the reservoir, such that the first non-trivial order of the mean value of the expansion of $U_t = \mathbb{1} - \mathrm{i}\int_0^t H_I^{(\mathrm{sl})}(t')U_{t'}\,\mathrm{d}t'$ on the state $\eta_0^{(\xi)} = \eta_0 \otimes \xi$ coincides with $I(t)$. The operator U_t defined by this integral equation is again called the *wave operator*.

The form of the generator follows now from an operation of normal ordering. In more detail, we start by defining the flux of an observable $\tilde{X} = X \otimes \mathbb{1}_r$, where $\mathbb{1}_r$ is the identity of the reservoir and X is an observable of the system, as $j_t(\tilde{X}) = U_t^\dagger \tilde{X} U_t$. Then, using the equation of motion for U_t and $U_t^\dagger$, we find that $\partial_t j_t(\tilde{X}) = \mathrm{i}U_t^\dagger\big[H_I^{(\mathrm{sl})}(t), \tilde{X}\big]U_t$. In order to compute the mean value of this equation on the state $\eta_0^{(\xi)}$, so to get rid of the reservoir operators, it is convenient to compute first the commutation relations between U_t and the limiting operators of the reservoir. At this stage the so-called time consecutive principle is used in a very heavy way to simplify the computation. This principle, which has been checked for many classes of physical models [3], states that if $\beta(t)$ is any of these limiting operators of the reservoir, then

$$[\beta(t), U_{t'}] = 0 \qquad \text{for all} \quad t > t'. \tag{A.5}$$

Using this principle and recalling that η_0 is annihilated by the limiting annihilation operators of the reservoir, it is now a simple exercise to compute $\langle \partial_t j_t(X)\rangle_{\eta_0^{(\xi)}}$ and, by means of the equation $\langle \partial_t j_t(X)\rangle_{\eta_0^{(\xi)}} = \langle j_t(L(X))\rangle_{\eta_0^{(\xi)}}$, to identify the form of the generator of the physical system.

Let us now consider the case in which $T > 0$. In this case the state of the reservoir is no longer given by φ_0. It is now convenient to use the so-called *canonical representation of* *thermal states* [3]. Using the same notation as section 2, any annihilator operator $a_{\vec{p},j}$ can be written as the following linear combination:

$$a_{\vec{p},j} = \sqrt{m(\vec{p})}c^{(a)}_{\vec{p},j} + \sqrt{n(\vec{p})}c^{(b),\dagger}_{\vec{p},j} \tag{A.6}$$

where $m(\vec{p})$ and $n(\vec{p})$ are the following two-points functions,

$$m(\vec{p}) = \omega_\beta\big(a_{\vec{p},j}a_{\vec{p},j\dagger}\big) = \frac{1}{1-\mathrm{e}^{-\beta\epsilon_{\vec{p}}}} \qquad n(\vec{p}) = \omega_\beta\big(a^\dagger_{\vec{p},j}a_{\vec{p},j}\big) = \frac{\mathrm{e}^{-\beta\epsilon_{\vec{p}}}}{1-\mathrm{e}^{-\beta\epsilon_{\vec{p}}}} \tag{A.7}$$

for our bosonic reservoir, if ω_β is a KMS state corresponding to an inverse temperature β. The operators $c^{(\alpha)}_{\vec{p},j}$ are assumed to satisfy the following commutation rules:

$$\big[c^{(\alpha)}_{\vec{p},j}, c^{(\gamma)\dagger}_{\vec{q},k}\big] = \delta_{jk}\delta_{\vec{p}\vec{q}}\delta_{\alpha\gamma} \tag{A.8}$$

while all the other commutators are trivial. Let, moreover, Φ_0 be the vacuum of the operators $c^{(\alpha)}_{\vec{p},j}$

$$c^{(\alpha)}_{\vec{p},j}\Phi_0 = 0 \qquad \forall\vec{p},\, j,\alpha.$$

Then it can immediately be checked that the results in (A.7) for the KMS state can be found, using these new variables, representing ω_β as the following vector state $\omega_\beta(\cdot) = \langle\Phi_0, \cdot\Phi_0\rangle$. With this GNS-like representation it is trivial to check that both the CCR and the two-point functions are easily recovered. This representation is also called in [3] the Fock–anti Fock representation because of the different sign in the free time evolution of the annihilation operators $c^{(a)}_{\vec{p},j}$ and $c^{(b)}_{\vec{p},j}$. Once this representation is introduced, all the same steps as for the situation with $T = 0$ can be repeated, and the expression for the generator can be deduced using exactly the same strategy.

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